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Tetrahedral Family Symmetry and the Neutrino Mixing Matrix

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Abstract

In a new application of the discrete non-Abelian symmetry A_4 using the canonical seesaw mechanism, a three-parameter form of the neutrino mass matrix is derived. It predicts the following mixing angles for neutrino oscillations: $\theta_{13} = 0$, $\sin^2\theta_{23} = 1/2$, and $\sin^2\theta_{12}$ close, but not exactly equal to 1/3, in one natural symmetry limit.

The symmetry group of the tetrahedron is also that of the even permutation of four objects, i.e. A_4. It is a non-Abelian finite subgroup of $SO(3)$ as well as $SU(3)$. It has twelve elements and four irreducible representations: $\underline{1}$, $\underline{1}'$, $\underline{1}''$, and $\underline{3}$. It has been shown to be useful in describing [1, 2, 3, 4, 5, 6, 7, 8] the family structure of quarks and leptons. In most previous applications, the lepton doublets (ν_i, l_i) are assigned to the $\underline{3}$ representation of A_4, whereas the charged-lepton singlets l_i^c are assigned to the three inequivalent one-dimensional representations $\underline{1}$, $\underline{1}'$, $\underline{1}''$. Here as in the two papers of Ref. [7], both (ν_i, l_i) and l_i^c are $\underline{3}$ instead.

Three heavy neutral fermion singelts N_i are assumed, transforming as $\underline{1}$, $\underline{1}'$, $\underline{1}''$ under A_4. [In the first paper of Ref. [7], they transform as $\underline{3}$; in the second, they are absent.] The multiplication rule $\underline{1}' \times \underline{1}'' = \underline{1}$ implies that the Majorana mass matrix of N_i invariant under A_4 is given by

$$\mathcal{M}_N = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & B \\ 0 & B & 0 \end{pmatrix}. \tag{1}$$

The multiplication rule

$$\underline{3} \times \underline{3} = \underline{1} + \underline{1}' + \underline{1}'' + \underline{3} + \underline{3} \tag{2}$$

allows the charged-lepton mass matrix to be diagonal by having three Higgs doublets transforming as $\underline{1}$, $\underline{1}'$, $\underline{1}''$, resulting in a diagonal $\mathcal{M}_l$ with

$$\begin{pmatrix} m_e \\ m_\mu \\ m_\tau \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} h_1 v_1 \\ h_2 v_2 \\ h_3 v_3 \end{pmatrix}, \tag{3}$$

where $\omega = \exp(2\pi i/3)$ and $v_{1,2,3}$ are the vacuum expectation values of these three Higgs doublets.

As for the Dirac mass matrix linking ν_i to N_j, three other Higgs doublets are assumed, transforming as $\underline{3}$ under A_4. [They are distinguished from the previous three Higgs doublets

by a discrete Z_2 symmetry.] Thus

$$\mathcal{M}_D = \begin{pmatrix} f_1 u_1 & f_2 u_1 & f_3 u_1 \\ f_1 u_2 & f_2 \omega u_2 & f_3 \omega^2 u_2 \\ f_1 u_3 & f_2 \omega^2 u_3 & f_3 \omega u_3 \end{pmatrix} = \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} f_1 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & f_3 \end{pmatrix}. \tag{4}$$

Using the canonical seesaw mechanism [9], the Majorana neutrino mass matrix is then given by

$$\mathcal{M}_\nu = \mathcal{M}_D \mathcal{M}_N^{-1} \mathcal{M}_D^T = \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{pmatrix} \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix} \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{pmatrix}, \tag{5}$$

where

$$a = f_1^2/A + 2 f_2 f_3/B, \quad b = f_1^2/A - f_2 f_3/B, \tag{6}$$

and $u_{1,2,3}$ are the vacuum expectation values of the second set of Higgs doublets which transform as $\underline{3}$ under A_4.

If $u_1 = u_2 = u_3 = u$, then a residual Z_3 symmetry exists, and the eigenvalues of $\mathcal{M}_\nu$ are simply $u^2(a+2b)$, $u^2(a-b)$, and $u^2(a-b)$. However, since the first eigenvalue corresponds to the eigenstate $(\nu_e + \nu_\mu + \nu_\tau)/\sqrt{3}$, this is not a realistic scenario. Consider now the case

$$u_2 = u_3 = u \neq u_1. \tag{7}$$

This makes $\mathcal{M}_\nu$ of the form advocated in Ref. [10] and results in $\theta_{13} = 0$ and $\theta_{23} = \pi/4$. Since $\theta_{13} = 0$ implies that CP is conserved in neutrino oscillations, the condition $u_2 = u_3$ should be considered "natural" in the sense that it is protected by a symmetry. Note that this alone does not imply $\theta_{23} = \pi/4$, which needs also A_4 for it to be true. [It certainly does not come from $\nu_\mu - \nu_\tau$ exchange as often suggested, because that would imply $\mu - \tau$ exchange as well, which cannot be sustained in the complete Lagrangian of the theory as a symmetry because $m_\mu \neq m_\tau$.]

Using the condition of Eq. (7), $\mathcal{M}_\nu$ of Eq. (5) can be rewritten as

$$\mathcal{M}_\nu = \begin{pmatrix} \lambda^2 a & \lambda b & \lambda b \\ \lambda b & a & b \\ \lambda b & b & a \end{pmatrix}. \tag{8}$$

In the basis ν_e, $(\nu_\mu + \nu_\tau)/\sqrt{2}$, and $(-\nu_\mu + \nu_\tau)/\sqrt{2}$, this becomes

$$\mathcal{M}_\nu = \begin{pmatrix} \lambda^2 a & \sqrt{2}\lambda b & 0 \\ \sqrt{2}\lambda b & a+b & 0 \\ 0 & 0 & a-b \end{pmatrix}, \tag{9}$$

yielding one exact eigenvalue and eigenstate:

$$m_3 = a - b, \quad \nu_3 = (-\nu_\mu + \nu_\tau)/\sqrt{2}. \tag{10}$$

In the submatrix spanning ν_e and $(\nu_\mu + \nu_\tau)/\sqrt{2}$, consider

$$\mathcal{M}_\nu \mathcal{M}_\nu^\dagger = \begin{pmatrix} |\lambda|^4|a|^2 + 2|\lambda|^2|b|^2 & \sqrt{2}\lambda(|b|^2 + a^*b + |\lambda|^2 ab^*) \\ \sqrt{2}\lambda^*(|b|^2 + ab^* + |\lambda|^2 a^*b) & |a+b|^2 + 2|\lambda|^2|b|^2 \end{pmatrix}. \tag{11}$$

The limit $|m_1|^2 = |m_2|^2$ is reached if

$$|a+b|^2 - |\lambda|^4|a|^2 = 0, \quad |b|^2 + a^*b + |\lambda|^2 ab^* = 0, \tag{12}$$

both of which are satisfied if $b = -a(1+|\lambda|^2)$. In this limit, $\Delta m^2_{sol} = 0$ and

$$\Delta m^2_{atm} \equiv |m_3|^2 - (|m_1|^2 + |m_2|^2)/2 = 2|a|^2(1-|\lambda|^4)(2+|\lambda|^2). \tag{13}$$

To obtain a nonzero Δm^2_{sol} and the value of θ_{12}, consider

$$b = -a(1 + |\lambda|^2 + \epsilon), \tag{14}$$

then

$$\begin{aligned} \Delta m^2_{sol} &\equiv |m_2|^2 - |m_1|^2 \\ &= |a|^2[|(\epsilon+\epsilon^*)|\lambda|^2 + |\epsilon|^2|^2 + 8|\lambda|^2|\epsilon^* + \epsilon|\lambda|^2 + |\epsilon|^2|^2]^{1/2}, \end{aligned} \tag{15}$$

and

$$\tan^2 2\theta_{12} = \frac{8|\lambda|^2|\epsilon^* + \epsilon|\lambda|^2 + |\epsilon|^2|^2}{|(\epsilon+\epsilon^*)|\lambda|^2 + |\epsilon|^2|^2}. \tag{16}$$

There are two natural limits of the parameter λ. (A) $\lambda = 1$ corresponds to $u_1 = u_2 = u_3 = u$, which is protected by a residual Z_3 symmetry as discussed already. (B) $\lambda = 0$

corresponds to $m_{\nu_e} = 0$ and the decoupling of ν_e from ν_μ and ν_τ, which is protected by a chiral U(1) symmetry. Hence values of λ near 1 and 0 will be considered from now on.

(A) For $|\lambda| \simeq 1$, ϵ is expected to be small compared to it in Eq. (14). In that case,

$$\Delta m^2_{sol} \simeq 2|a|^2|\lambda|[(Re\epsilon)^2(2+|\lambda|^2)(1+2|\lambda|^2) + 2(Im\epsilon)^2(1-|\lambda|^2)^2]^{1/2}, \tag{17}$$

and

$$\tan^2 2\theta_{12} \simeq 8\left[\left(\frac{1+|\lambda|^2}{2|\lambda|}\right)^2 + \frac{(Im\epsilon)^2}{(Re\epsilon)^2}\left(\frac{1-|\lambda|^2}{2|\lambda|}\right)^2\right]. \tag{18}$$

This means that $|\tan 2\theta_{12}| > 2\sqrt{2}$, or equivalently $\sin^2\theta_{12} > 1/3$, to be compared with the current experimental fit of $\sin^2\theta_{12} = 0.31 \pm 0.03$.

Using the typical experimental values

$$\Delta m^2_{atm} = 2.5 \times 10^{-3}\ \text{eV}^2, \quad \Delta m^2_{sol} = 8.0 \times 10^{-5}\ \text{eV}^2, \tag{19}$$

and assuming ϵ to be real, its value and those of $\sin^2\theta_{12}$ and $|\lambda^2 a|$ are given in Table 1. It shows that $\sin^2\theta_{12}$ is very near 1/3 and cannot be distinguished in practice from being

Table 1: Values of $\sin^2\theta_{12}$, ϵ, and $|\lambda^2 a|$ as functions of $|\lambda|$.

$\lvert\lambda\rvert$	$\sin^2\theta_{12}$	ϵ	$\lvert\lambda^2 a\rvert$
0.7	0.342	0.027	0.013 eV
0.8	0.337	0.020	0.018 eV
0.9	0.334	0.011	0.029 eV
1.0	0.333	–	–
1.1	0.334	0.014	0.035 eV
1.2	0.336	0.032	0.026 eV
1.3	0.338	0.055	0.023 eV
1.4	0.341	0.082	0.021 eV

exactly 1/3 [11], as in some models. The last column corresponds to the expected value of the effective neutrino mass measured in neutrinoless double beta decay.

(B) For $|\lambda| \simeq 0$, consider $|\epsilon|$ also to be of order $|\lambda|$, then

$$\Delta m^2_{atm} \simeq 4|a|^2, \tag{20}$$

$$\Delta m^2_{sol} \simeq |a|^2|\epsilon|\sqrt{|\epsilon|^2 + 8|\lambda|^2}, \tag{21}$$

$$\tan^2 2\theta_{12} \simeq 8|\lambda|^2/|\epsilon|^2. \tag{22}$$

In this case, $|a| = 0.025$ eV, and $\sin^2\theta_{13} < 1/3$ can be obtained for $|\lambda| < |\epsilon|$. Suppose it is fixed at 0.31, then $|\lambda| = 0.19$, $|\epsilon| = 0.22$, and $|\lambda^2 a| = 9.0 \times 10^{-4}$ eV.

In conclusion, it has been shown in this paper that a new application of the non-Abelian discrete symmetry A_4 in the context of the canonical seesaw mechanism is successful in obtaining a realistic neutrino mixing matrix with $\theta_{13} = 0$, $\theta_{23} = \pi/4$, and a prediction of $\sin^2\theta_{12}$ very near 1/3 in a particular symmetry limit. As Eq. (13) shows, the normal (inverted) hierarchy of neutrino masses is obtained for $|\lambda|$ less (greater) than 1. Typical values of the effective neutrino mass measured in neutrinoless double beta decay are given in Table 1.

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